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A compactness result for Kähler Ricci solitons

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Abstract

In this paper we prove a compactness result for compact Kähler Ricci gradient shrinking solitons. If (M_i, g_i) is a sequence of Kähler Ricci solitons of real dimension $n \geqslant 4$, whose curvatures have uniformly bounded $L^{n/2}$ norms, whose Ricci curvatures are uniformly bounded from below and $\mu(g_i, 1/2) \geqslant A$ (where μ is Perelman's functional), there is a subsequence (M_i, g_i) converging to a compact orbifold (M_∞, g_∞) with finitely many isolated singularities, where g_∞ is a Kähler Ricci soliton metric in an orbifold sense (satisfies a soliton equation away from singular points and smoothly extends in some gauge to a metric satisfying Kähler Ricci soliton equation in a lifting around singular points).
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1. Introduction

Let $g(t)$ be a Kähler Ricci flow on a compact, Kähler manifold M,

$$\frac{d}{dt} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} = u_{i\bar{j}}. \tag{1}$$

Very special solutions of (1) are Kähler Ricci solitons.

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Definition 1. A Kähler Ricci soliton is a solution of (1) that moves by a one parameter family of biholomorphisms (which are induced by a holomorphic vector field). If this vector field comes from a gradient of a function, we have a gradient Kähler Ricci soliton. In particular, the equation of a gradient Kähler Ricci soliton g is

$$g_{i\bar{j}} - R_{i\bar{j}} = u_{i\bar{j}}, \tag{2}$$

with $u_{ij} = u_{\bar{i}\bar{j}} = 0$.

Soliton type solutions are very important ones, since they usually appear as blow up limits of the Ricci flow and therefore understanding this kind of solutions helps us understand singularities that the flow can develop either in a finite or infinite time. Solitons are generalizations of Einstein metrics (notice that if a function u in (2) is constant, we exactly get a Kähler–Einstein metric). Compactness theorem for Kähler–Einstein metrics has been considered and proved by Anderson [1], Bando, Kasue and Nakajima [2], and Tian [10]. They all proved that if we start with a sequence of Kähler–Einstein metrics which have uniformly bounded $L^{n/2}$ norms of a curvature, uniformly bounded Ricci curvatures and diameters, and if $\mathrm{Vol}_{g_i} M_i \geqslant V$ for all i, then there exists a subsequence (M_i, g_i) and a compact Kähler Einstein orbifold (M_∞, g_∞) with a finite set of singularities, so that $(M_i, g_i) \to (M_\infty, g_\infty)$ smoothly away from singular points (in the sense of Gromov–Cheeger convergence). Moreover, for each singular point there is a neighbourhood and a gauge so that a lifting of a singular metric in that gauge has a smooth extension over the origin. The extended smooth metric is also Kähler–Einstein metric. It is natural to expect that something similar holds in the case of Kähler Ricci solitons.

Our goal in this paper is to prove the compactness result for the Kähler Ricci solitons, that is, we want to prove the following theorem.

Theorem 2. *Let (M_i, g_i) be a sequence of Kähler Ricci solitons of real dimension $n \geqslant 6$, with $c_1(M_i) > 0$,*

$$\frac{d}{dt} g_i(t) = g_i(t) - \mathrm{Ric}\big(g_i(t)\big) = \partial\bar{\partial} u_i(t), \tag{3}$$

with $\nabla_j \nabla_k u_i = \bar{\nabla}_j \bar{\nabla}_k u_i = 0$, such that

(a) $\int_{M_i} |\mathrm{Rm}|^{n/2}\, dV_{g_i} \leqslant C_1$,
(b) $\mathrm{Ric}(g_i) \geqslant -C_2$,
(c) $A \leqslant \mu(g_i, 1/2)$,

for some uniform constants C_1, C_2, A, independent of i. Then there exists a subsequence (M_i, g_i) converging to $(Y, \bar{g})$, where Y is an orbifold with finitely many isolated singularities and $\bar{g}$ is a Kähler Ricci soliton in an orbifold sense (*see a definition below*).

We have a similar result in the case $n = 4$, but because of a different treatment we will state it separately.

Theorem 3. *Let (M_i, g_i) be a sequence of complex 2-dimensional Kähler Ricci solitons with $c_1(M_i) > 0$, satisfying*

$$\frac{d}{dt} g_i(t) = g_i(t) - \operatorname{Ric}\bigl(g_i(t)\bigr) = \partial\bar{\partial} u_i(t), \tag{4}$$

with $\nabla_j \nabla_k u_i = \bar{\nabla}_j \bar{\nabla}_k u_i = 0$, such that

(a) $|\operatorname{Ric}(g_i)| \leqslant C_1$,
(b) $A \leqslant \mu(g_i, 1/2)$,

for some uniform constants C_1, A, independent of i. Then there exists a subsequence (M_i, g_i) converging to $(Y, \bar{g})$, where Y is an orbifold with finitely many isolated singularities and $\bar{g}$ is a Kähler Ricci soliton in an orbifold sense.

First of all notice that in Theorem 3 we have one condition less. That is because in the case $n = 4$ the condition (a) of Theorem 2 is automatically fulfilled (a consequence of a Gauss–Bonett formula for surfaces). The approach that we will use to prove Theorem 2 is based on Sibner's idea for treating the isolated singularities for the Yang–Mills equations which requires $n > 4$. To prove Theorem 3, we will use the techniques developed by Uhlenbeck in [12] to treat the Yang–Mills equation, and then later used by Tian in [10] to deal with an Einstein equation.

Definition 4. Let (M_i, g_i) be a sequence of Kähler manifolds, of real dimension n (n is taken to be even). We will say that (M_i, g_i) converge to an orbifold (M_∞, g_∞) with finitely many isolated singularities $p_1, \dots, p_N$, where g_∞ is a *Kähler Ricci soliton in an orbifold sense*, if

(a) For any compact subset $K \subset M_\infty \backslash \{p_1, \dots, p_N\}$ there are compact sets $K_i \subset M_i$ and diffeomorphisms $\phi_i : K_i \to K$ so that $(\phi_i^{-1})^* g_i$ converge to g_∞ uniformly on K and $\phi_{i*} \circ J_i \circ (\phi_i^{-1})_*$ converge to J_∞ uniformly on K, where J_i, J_∞ are the almost complex structures of M_i, M_∞, respectively.
(b) For every p_i there is a neighbourhood U_i of p_i in M_∞ that is covered by a ball Δ_r in $\mathrm{C}^{n/2}$ with the covering group isomorphic to a finite group in $U(2)$. Moreover, if $\pi_i : \Delta_r \to U_i$ is the covering map, there is a diffeomorphism ψ of Δ_r so that $\phi^* \pi_i^* g_\infty$ smoothly extends to a Kähler Ricci soliton C^∞-metric on Δ_r in $\mathrm{C}^{n/2}$ with respect to the standard complex structure.

We will call (M_∞, g_∞) a *generalized Kähler Ricci soliton.*

The outline of the proof of Theorem 2 is as follows.

1. Obtaining the ϵ-regularity lemma for Kähler Ricci solitons (the analogue of the existing one for Einstein metrics) which says that a smallness of the $L^{n/2}$ norm of a curvature implies a pointwise bound on the curvature.
2. Combining the previous step together with a uniform $L^{n/2}$ bound on the curvatures of solitons in our sequence yields a convergence of a subsequence of our solitons to a topological orbifold (M_∞, g_∞) with finitely many isolated singularities. The metric g_∞ satisfies the Kähler Ricci soliton equation away from singular points.

3. Using Moser iteration argument (as in [2,10]) we can show the uniform boundness of $|\mathrm{Rm}(g_\infty)|$ on $M_\infty \backslash \{\text{singular points}\}$.
4. By the similar arguments as in [1,2,10] we can show that g_∞ extends to a C^0 orbifold metric on M_∞.
5. Using that Ricci potentials of metrics g_i in our sequence are the minimizers of Perelman's functional $\mathcal{W}$, henceforth satisfying the elliptic equation, and using harmonic coordinates around the orbifold points we can show that g_∞ extends to a C^∞ orbifold metric on M_∞. A lifting of g_∞ above orbifold points is a smooth metric, satisfying a Kähler Ricci soliton equation in the covering space.

Due to Perelman, instead of assuming uniform bounds on diameters and volume noncollapsing constant it is enough to assume condition (c) in Theorem 2 (see the next section for more details). In [6] Perelman has introduced Perelman's functional

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M e^{-f}\left[\tau\left(R + |\nabla f|^2\right) + f - n\right] dV_g,$$

under the constraint

$$(4\pi\tau)^{-n/2} \int_M e^{-f}\, dV_g = 1. \tag{5}$$

He also defined $\mu(g,\tau) = \inf \mathcal{W}(g,\cdot,\tau)$, where inf is taken over all functions satisfying the constraint (5).

The compactness theorem for Kähler Einstein manifolds has been established in [1,2,10]. Almost the same proof of Theorem 2 yields a generalization of very well known compactness theorem in Kähler Einstein case.

Theorem 5. *Let (M_i, g_i) be a sequence of Kähler Ricci solitons with $c_1(M_i) > 0$, such that the following holds*:

(a) $\int_{M_i} |\mathrm{Rm}|^{n/2} dV_{g_i} \leqslant C_1$,
(b) $\mathrm{Ric}(g_i) \geqslant -C_2$ *for* $n \geqslant 6$ *and* $|\mathrm{Ric}(g_i)| \leqslant C_2$ *for* $n = 4$,
(c) $\mathrm{Vol}_{g_i}(B_{g_i}(x,r)) \geqslant \kappa r^n$,
(d) $\mathrm{diam}(M_i, g_i) \leqslant C_3$,

for some uniform constants C_1, C_2, κ, C_3, independent of i. Then there exists a subsequence (M_i, g_i) converging to (M_∞, g_∞), where M_∞ is an orbifold with finitely many isolated singularities and g_∞ is a Kähler Ricci soliton in an orbifold sense.

Due to Perelman's results for the Kähler Ricci flow (see [7]), and due to the observation of Klaus Ecker (that will be discussed in the following section), that was communicated to the second author by Berhnard List we have the following corollary.

Corollary 6. *Theorems* 2 *and* 3 *are equivalent to Theorem* 5 *for dimensions* $n \geqslant 6$ *and* $n = 4$, *respectively.*

The proof of Theorem 5 is essentially the same as that of Theorem 2. The only difference is that we do not have a uniform lower bound on Perelman's functional μ, so we have imposed uniform bounds on diameters, volume noncollapsing and a euclidean volume growth, which are also either implied or given, in the case we start with a sequence of Kähler Einstein manifolds.

2. Perelman's functional $\mu(g, 1/2)$

We can normalize our solitons, so that $\mathrm{Vol}_{g_i}(M) = 1$. In order to prove convergence we need some sort of ϵ-regularity lemma (an analogue of ϵ-regularity lemma for Einstein manifolds, adopted to the case of Kähler Ricci solitons). We will use Moser iteration argument to get a quadratic curvature decay away from curvature concentration points. Define

$$\mathcal{S}_n(C_2, A, V) = \{\text{Kähler Ricci solitons } g \mid \mu(g,\tau) \geqslant A, \text{ for all } \tau \in (0,1) \text{ and } \mathrm{Ric}(g) \geqslant -C_2\}.$$

Kähler Ricci shrinking solitons g_i from Theorem 2 are in $\mathcal{S}_n$. Due to Perelman we know that the scalar curvature of a solution $g(t)$ satisfying only the first defining condition of $\mathcal{S}_n$ is uniformly bounded along the flow. Perelman also showed this implies $g(t)$ is κ-noncollapsed, where $\kappa = \kappa(A)$. In particular, these bounds are uniform for all elements in $\mathcal{S}_n$ and

$$|R(g_i)| \leqslant \tilde{C}, \tag{6}$$

for all i.

Lemma 7. *If* $\frac{d}{dt}g(t) = g(t) - \mathrm{Ric}(g(t)) = \partial\bar{\partial}u$ *is a shrinking gradient Kähler Ricci soliton, then* $u(t)$ *is a minimizer of Perelman's functional* $\mathcal{W}$ *with respect to metric* $g(t)$.

Proof. Let $f(0)$ be a minimizer of $\mathcal{W}$ with respect to metric $g(0)$. Let $\phi(t)$ be a 1-parameter family of biholomorphisms that come from a holomorphic vector field $\nabla u(t)$, such that $g(t) = \phi(t)^* g(0)$. Function $f(t) = \phi^* f(0)$ is a minimizer of $\mathcal{W}$ with respect to metric $g(t)$ since $e^{-f(t)}\,dV_t = dm = \text{const}$ and since

$$\mu(g(t),\tau) \leqslant \mathcal{W}(g(t), f(t), \tau) = \mathcal{W}(g(0), f(0), \tau) = \mu(g(0),\tau) \leqslant \mu(g(t),\tau),$$

where the last inequality comes from Perelman's monotonicity for the Ricci flow. We have that

$$\mathcal{W}(g(0), f(0), \tau) = \mathcal{W}(g(t), f(t), \tau),$$

and therefore $\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau) = 0$. On the other hand, $e^{-f(t)}\,dV_t = dm = \text{const}$ and by Perelman's monotonicity formula

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{W}(g(t), f(t), \tau) \\ &= (4\pi\tau)^{-n/2}\int_M e^{-f(t)}\left|R_{i\bar{j}} + f_{i\bar{j}} - g_{i\bar{j}}/(2\tau)\right|^2 dV_t, \end{aligned}$$

which implies $R_{i\bar{j}} + f_{i\bar{j}} - g_{i\bar{j}}/(2\tau) = 0$ on M, that is $\Delta f(t) = n/2 - R = \Delta u(t)$ and since M is compact, $f(t) = u(t)$ (both functions satisfy the same integral normalization condition $\int_M e^{-f(t)}\, dV_t = \int_M e^{-u(t)}\, dV_t = (4\pi\tau)^{n/2}$). □

Take $\tau = 1/2$. We have our sequence of Kähler Ricci solitons (M, g_i) which defines a sequence of Kähler Ricci flows $\{g_i(t)\}$, where $g_i \in \mathcal{S}_n(C_2, A, V)$ and u_i is a Ricci potential for g_i. The previous lemma tells us that every u_i is a minimizer of $\mathcal{W}(g_i, \cdot, 1/2)$ and therefore satisfies

$$2\Delta u_i - |\nabla u_i|^2 + R(g_i) + u_i - n = \mu(g_i, 1/2),$$

which implies $u_i = |\nabla u_i|^2 + \mu(g_i, 1/2) + R(g_i) \geqslant -\tilde{C}$, by (6) and condition (c) in Theorem 2. Since we have a uniform lower bound on u_i, as in [7] we have that

$$u_i(y,t) \leqslant C\, \mathrm{dist}_{it}^2(x_i, y) + C,$$
$$|\nabla u_i| \leqslant C\, \mathrm{dist}_{it}(x_i, y) + C,$$

for a uniform constant C, where $u_i(x_i, t) = \min_{y \in M_i} u_i(y,t)$ and $\mathrm{dist}_{it}(x, y)$ is a distance between points x and y, measured in metrics $g_i(t)$. In order to prove that $|u_i(t)|_{C^1} \leqslant C$ for a uniform constant C, it is enough to show that the diameters of $(M_i, g_i(t))$ are uniformly bounded. Since we have (a), (b), (c) and since $\mathrm{Vol}_{g_i}(M) = 1$ for all i, by the same proof as in [7] we can show that the diameters of $(M_i, g_i(t))$ are indeed uniformly bounded. Therefore, there are uniform constants C and κ such that for all i,

1. $|u_i|_{C^1} \leqslant C$,
2. $\mathrm{diam}(M_i, g_i) \leqslant C$,
3. $|R(g_i)| \leqslant C$,
4. (M_i, g_i) is κ-noncollapsed.

This immediately implies that Theorems 2 and 3 imply Theorem 5 for dimensions $n \geqslant 6$ and $n = 4$, respectively.

A uniform lower bound on Ricci curvatures, a uniform volume noncollapsing condition and a uniform upper bound on diameters give us a uniform upper bound on Sobolev constants of (M_i, g_i), that is, there is a uniform constant S so that for every i and for every Lipschitz function f on M_i,

$$\left\{ \int_{M_i} (f\eta)^{\frac{2n}{n-2}}\, dV_{g_i} \right\}^{\frac{n-2}{n}} \leqslant S \int_{M_i} \left|\nabla(\eta f)\right|^2 dV_{g_i}, \tag{7}$$

where η is a cut off function on M_i.

A uniform lower bound on Ricci curvatures implies the existence of a uniform constant V such that

$$\mathrm{Vol}_{g_i}\big(B_{g_i}(p, r)\big) \leqslant V r^n, \tag{8}$$

for all i, $p \in M$ and all $r > 0$. By Bishop–Gromov volume comparison principle we have

$$\mathrm{Vol}_{g_i}\big(B_{g_i}(p, r)\big) \leqslant V_{-C_2}(r) \frac{\mathrm{Vol}_{g_i}(B_{g_i}(p, \delta))}{V_{-C_2}(\delta)},$$

which by letting $\delta \to 0$ yields

$$\mathrm{Vol}_{g_i}\big(B_{g_i}(p,r)\big) \leqslant w_n V_{-C_2}(r) = r^n w_n V_{-C_2 r}(1),$$

where w_n is a volume of a Euclidean unit ball and $V_{-C_2}(r)$ is a volume of a ball of radius r in a simply connected space of constant sectional curvature $-C_2$. The term on the right-hand side of the previous estimate is bounded by Vr^n, for a uniform constant V, since $\mathrm{diam}(M, g_i) \leqslant D$ and therefore $0 \leqslant C_2 r \leqslant DC_2$.

Ecker's observation that finishes the proof of Corollary 6 is as follows.

Lemma 8 *(Ecker). There is a lower bound on $\mu(g,\tau)$ in terms of a Sobolev constant C_S for g, that is,*

$$\mu(g,\tau) \geqslant -C(n)\big(1 + \ln C_S(g) + \ln\tau\big) + \tau \inf_M R(g).$$

Proof. Let f be a minimizer for $\mathcal{W}$, and let $u = \phi^2 = (4\pi\tau)^{-n/2}e^{-f}$. Then,

$$\begin{aligned}\mu(g,\tau) &\geqslant (4\pi\tau)^{-n/2}\int_M \big(4\tau|\nabla\phi|^2 - \phi^2\ln\phi^2\big)\,dV + \tau\inf_M R(g) - c(n)(1+\ln\tau)\\ &= I + \tau\inf_M R(g) - c(n)(1+\ln\tau).\end{aligned} \tag{9}$$

Rescale $g_\tau = g/(4\tau)$, $\phi_\tau = (4\tau)^{n/2}\phi$. Then,

$$I = \int_M \big(|\nabla\phi_\tau|_\tau^2 - \phi_\tau^2\ln\phi_\tau^2\big)\,dV_\tau,$$

with $\int_M \phi_\tau^2\,dV_\tau = 1$. The usual Sobolev inequality (with constant C_S) implies a logarithmic Sobolev inequality

$$\int_M \big(|\nabla w|^2 - w^2\ln w^2\big)\,dV_g \geqslant C(n)\big(1+\ln C_S(g)\big),$$

for every $w \geqslant 0$ and $\int_M w^2\,dV_g = 1$. Apply the previous inequality to $w = \phi_\tau$ and to g_τ (note that $C_S(g_\tau) \leqslant (1+2\sqrt{\tau})C_S(g)$, which implies

$$I \geqslant -C(n)\big(1 + \ln\big((1+2\sqrt{\tau})C_S(g)\big)\big). \qquad \square$$

Conditions that we are imposing in Theorems 2 and 3 are enough to obtain a uniform upper bound on Sobolev constants $C_S(g_i)$. This together with (9) gives a uniform lower bound on $\mu(g_i,\tau)$.

3. ϵ-regularity lemma for Kähler Ricci solitons

In this section we will establish ϵ-regularity lemma for Kähler Ricci solitons.

By Bochner–Weitzenbock formulas we have

$$\Delta|\mathrm{Rm}|^2 = -2\langle\Delta\mathrm{Rm},\mathrm{Rm}\rangle + 2|\nabla\mathrm{Rm}|^2 - \big\langle Q(\mathrm{Rm}),\mathrm{Rm}\big\rangle, \tag{10}$$

where $Q(\mathrm{Rm})$ is quadratic in Rm. The Laplacian of a curvature tensor in the Kähler case reduces to

$$\Delta R_{i\bar{j}k\bar{l}} = \nabla_i\nabla_{\bar{l}}R_{\bar{j}k} + \nabla_{\bar{j}}\nabla_k R_{i\bar{l}} + S_{i\bar{j}k\bar{l}},$$

where $S(\mathrm{Rm})$ is quadratic in Rm. In the case of a soliton metric $g \in \mathcal{S}_n(C_2, A, V)$ on M, that satisfies $g_{i\bar{j}} - R_{i\bar{j}} = \partial_i\bar{\partial}_j u$, by commuting the covariant derivatives, we get

$$\begin{aligned}\Delta R_{i\bar{j}k\bar{l}} &= u_{\bar{j}k\bar{l}i} + u_{i\bar{l}k\bar{j}} + S_{i\bar{j}k\bar{l}}\\ &= u_{\bar{j}\bar{l}ki} + \nabla_i(R_{\bar{j}k\bar{l}m}u_m) + u_{ik\bar{l}\bar{j}} + \nabla_{\bar{j}}(R_{i\bar{l}k\bar{m}}u_{\bar{m}}) + S_{i\bar{j}k\bar{l}}\\ &= \nabla_i(R_{\bar{j}k\bar{l}m})u_m + \nabla_{\bar{j}}(R_{i\bar{l}k\bar{m}})u_{\bar{m}} + S_{i\bar{j}k\bar{l}}\\ &= \nabla\mathrm{Rm} * \nabla u + S_{i\bar{j}k\bar{l}},\end{aligned} \tag{11}$$

where we have effectively used the fact that $u_{ij} = u_{\bar{i}\bar{j}} = 0$ and $A * B$ denotes any tensor product of two tensors A and B when we do not need precise expressions. By using that $|u|_{C^1} \leqslant C$ on M and identities (10) and (11) we get

$$\Delta|\mathrm{Rm}|^2 \geqslant -C|\nabla\mathrm{Rm}||\mathrm{Rm}| + 2|\nabla\mathrm{Rm}|^2 - C|\mathrm{Rm}|^3.$$

By interpolation inequality we have

$$\begin{aligned}\Delta|\mathrm{Rm}|^2 &\geqslant (2-\theta)|\nabla\mathrm{Rm}|^2 - C(\theta)|\mathrm{Rm}|^2 - C|\mathrm{Rm}|^3\\ &\geqslant (2-\theta)\big|\nabla|\mathrm{Rm}|\big|^2 - C(\theta)|\mathrm{Rm}|^2 - C|\mathrm{Rm}|^3.\end{aligned}$$

We will see later how small θ we will take. Also,

$$\Delta|\mathrm{Rm}|^2 = 2\Delta|\mathrm{Rm}||\mathrm{Rm}| + 2\big|\nabla|\mathrm{Rm}|\big|^2,$$

and therefore,

$$\Delta|\mathrm{Rm}||\mathrm{Rm}| \geqslant -\theta/2\big|\nabla|\mathrm{Rm}|\big|^2 - C(\theta)|\mathrm{Rm}|^2 - C|\mathrm{Rm}|^3. \tag{12}$$

Denote by $u = |\mathrm{Rm}|$. Then,

$$u\Delta u \geqslant -\theta/2|\nabla u|^2 - C(\theta)u^2 - Cu^3. \tag{13}$$

We can now prove the ϵ-regularity theorem for shrinking gradient Kähler Ricci solitons.

Theorem 9. *Let g be a Kähler Ricci soliton as above. Then there exist constants $C = C(n,\kappa)$ and $\epsilon = \epsilon(n,\kappa)$ so that for $r < (\epsilon/V)^{1/n} = r_0$ if*

$$\int\limits_{B(p,2r)} |\mathrm{Rm}|^{n/2}\, dV_g < \epsilon, \tag{14}$$

then

$$\sup_{B(p,r/2)} \left|\mathrm{Rm}(g)\right|(x) \leqslant \frac{C}{r^2}. \tag{15}$$

Proof. It simplifies matters if we assume $r = 1$. We may assume that, since (12), condition (14) and Claim (15) are all scale invariant. We can start with r so that $\mathrm{Vol}\, B_g(p,2r)$ is sufficiently small (it will become clearer from the proceeding discussion), then rescale our metric so that $r = 1$. In the rescaled metrics, $\mathrm{Vol}_g\, B(p,2)$ is small and due to the invariance property of $L^{n/2}$ norm of $|\mathrm{Rm}|$, $\int_{B_g(p,2)} |\mathrm{Rm}|^{n/2} < \epsilon$ (we are using g to denote a rescaled metric as well).

We will first prove $L^{q'}$ bound on Rm, for some $q' > n/2$. Let S be a uniform Sobolev constant, V as in (8).

Lemma 10. *There is $q' > n/2$ so that $\int_{B_g(p,1)} |\mathrm{Rm}|^{q'}\, dV_g \leqslant C$, where $C = C(n,q',V,S)$.*

Proof. Let ϕ be a nonnegative cut off function that we will choose later and $q \geqslant 2$. Multiply (13) by $\phi^2 u^{q-2}$ and integrate it over M.

$$\begin{aligned} &\theta/2 \int \phi^2 |\nabla u|^2 u^{q-2} + C(\theta) \int \phi^2 \left(u^q + u^{q+1}\right) \\ &\quad \geqslant \int \phi^2 u^{q-1}(-\Delta u) \geqslant 4(q-1)q^{-2} \int \left|\phi \nabla\left(u^{q/2}\right)\right|^2 + 4q^{-1} \int \phi u^{q/2} \left\langle \nabla\phi, \nabla\left(u^{q/2}\right)\right\rangle. \end{aligned} \tag{16}$$

We can write $\frac{4}{q^2} \int \phi^2 |\nabla u^{q/2}|^2$ for $\int \phi^2 |\nabla u|^2 u^{q-2}$, apply Schwartz and interpolation inequalities to the second term on the right-hand side of (16) and get

$$\begin{aligned} 2(q-1)q^{-2} \int \left|\phi \nabla u^{q/2}\right|^2 \leqslant \frac{2\theta}{q^2} \int \phi^2 \left|\nabla u^{q/2}\right|^2 + C(\theta) \int \phi^2 \left(u^q + u^{q+1}\right) \\ + 2(q-1)^{-1} \int |\nabla\phi|^2 u^q. \end{aligned} \tag{17}$$

Choose $0 < \theta < q-1$ (in particular we can choose $\theta = (q-1)/2$). Then

$$(q-1)q^{-2} \int \left|\phi \nabla u^{q/2}\right|^2 \leqslant C \int \phi^2 \left(u^q + u^{q+1}\right) + 2(q-1)^{-1} \int |\nabla\phi|^2 u^q, \tag{18}$$

where $C = C(n)$. Using the Sobolev inequality (let $\gamma = n/(n-2)$) we obtain

$$\begin{aligned}\left\{\int \left|\phi u^{q/2}\right|^{2\gamma}\right\}^{1/\gamma} &\leqslant \tilde{C}\left\{q^2(q-1)^{-1}\int \phi^2\left(u^{q+1}+u^q\right)+\left(2q^2(q-1)^{-2}+1\right)\int |\nabla\phi|^2u^q\right\}\\ &\leqslant \tilde{C}q^2(q-1)^{-1}\left\{\left(\int\limits_{\operatorname{supp}\phi} u^{n/2}\right)^{2/n}+\left(\int\limits_{\operatorname{supp}\phi} dV\right)^{2/n}\right\}\left\{\int \left|\phi u^{q/2}\right|^{2\gamma}\right\}^{1/\gamma}\\ &\quad+\tilde{C}\left(2q^2(q-1)^{-2}+2\right)\int |\nabla\phi|^2u^q,\end{aligned}$$

where $\tilde{C}=\tilde{C}(n,S,V)$. Take $\epsilon=\tilde{C}^{-1}q^{-2}(q-1)/4$ and let ϕ be a cut off function with compact support in $B_g(p,2)$, equal to 1 on $B_g(p,1)$ and such that $|\nabla\phi|\leqslant C$. Take $q=n/2$. Then we get the following estimate:

$$\left\{\int \left|\phi u^{q'}\right|\right\}^{1/\gamma}\leqslant \bar{C}_1,$$

that is,

$$\int\limits_{B_g(p,1)} u^{q'}\,dV_g\leqslant \bar{C}, \tag{19}$$

where $\bar{C}=\bar{C}(C_2,A,V,n)$ and $q'=q\gamma=n\gamma/2>n/2$. $\square$

Let q' be as in Lemma 10. Take some $\beta>3/2$, let ϕ be a function with compact support in $B_g(p,1)$, equal to 1 on $B_g(p,1/2)$ and choose $\theta=1/2$ in (17). Then the estimate becomes

$$\begin{aligned}\int \left|\phi\nabla u^{\beta/2}\right|^2 &\leqslant \tilde{C}\left(\frac{\beta^2}{2\beta-3}\right)\left(\int\phi^2u^\beta+\int\phi^2u^{q+1}\right)+2\frac{\beta^2}{(\beta-1)(2\beta-3)}\int|\nabla\phi|^2u^\beta\\ &\leqslant \tilde{C}\left(\frac{\beta^2}{\beta-1}\right)\left(\int\phi^2u^\beta+\int\phi^2u^{q+1}\right)+2\frac{\beta^2}{(\beta-1)^2}\int|\nabla\phi|^2u^\beta\\ &\leqslant \tilde{C}_1(\beta+1)\left(\int\phi^2u^\beta+\int\phi^2u^{\beta+1}\right)+C\int|\nabla\phi|^2u^\beta.\end{aligned}$$

Furthermore, by (19) we have

$$\int\phi^2u^{\beta+1}\leqslant\left(\int\limits_{B_g(p,1)} u^{q'}\right)^{1/q'}\left(\int\left|\phi u^{\beta/2}\right|^{2\gamma'}\right)^{1/\gamma'}\leqslant\bar{C}_2\left(\int\left|\phi u^{\beta/2}\right|^{2\gamma'}\right)^{1/\gamma'},$$

with $\gamma'=q'/(q'-1)$. By interpolation inequality, with $2^*=2n/(n-2)>2q'/(q'-1)>2$, since $q'>n/2$, we have

$$\left\|\phi u^{\beta/2}\right\|_{L^{\frac{2q'}{q'-1}}}\leqslant\eta\left\|\phi u^{\beta/2}\right\|_{L^{2^*}}+C(n,q')\eta^{-\frac{n}{2q'-n}}\left\|\phi u^{\beta/2}\right\|_{L^2},$$

for any small $\eta > 0$. By Sobolev inequality,

$$\left\|\phi u^{\beta/2}\right\|_{L^{\frac{2q'}{q'-1}}} \leqslant S\eta\left\|\nabla\left(\phi u^{\beta/2}\right)\right\|_{L^2} + C(n,q')\eta^{-\frac{n}{2q'-n}}\left\|\phi u^{\beta/2}\right\|_{L^2}.$$

All this yields

$$\begin{aligned}\int_M \left|\nabla\left(\phi u^{\beta/2}\right)\right|^2 &\leqslant \tilde{C}_1(1+\beta)\int \phi^2 u^\beta + (C+1)\int |\nabla\phi|^2 u^\beta + \tilde{C}_1\bar{C}_2(\beta+1)\left\{\int \left|\phi u^{\beta/2}\right|^{2\gamma'}\right\}^{1/\gamma'} \\ &\leqslant \tilde{C}_1(1+\beta)\int \phi^2 u^\beta + (C+1)\int |\nabla\phi|^2 u^\beta + \tilde{C}_1(\beta+1)\bar{C}_2\eta^2\left\|\phi u^{\beta/2}\right\|^2_{L^{2*}} \\ &\quad + C(n,q')^2\eta^{-\frac{2n}{2q'-n}}\left\|\phi u^{\beta/2}\right\|^2_{L^2} \\ &\leqslant \tilde{C}_1(1+\beta)\int \phi^2 u^\beta + (C+1)\int |\nabla\phi|^2 u^\beta + \tilde{C}_1(\beta+1)\bar{C}_2 S\left\|\nabla\left(\phi u^{\beta/2}\right)\right\|^2_{L^2} \\ &\quad + C(n,q')^2\eta^{-\frac{2n}{2q'-n}}\left\|\phi u^{\beta/2}\right\|^2_{L^2}.\end{aligned}$$

Choose $\eta^2 = \frac{1}{3\tilde{C}_1(\beta+1)\bar{C}_2 S}$. Then,

$$\begin{aligned}\int \left|\nabla\left(\phi u^{\beta/2}\right)\right|^2 &\leqslant C_3\int |\nabla\phi|^2 u^\beta + C_4\left(1+\beta+(1+\beta)^{\frac{2n}{2q'-n}}\right)\int \phi^2 u^\beta \\ &\leqslant C_5(1+\beta)^\alpha \int \left(|\nabla\phi|^2+\phi^2\right)u^\beta,\end{aligned}$$

where α is a positive number depending only on n and q'. Sobolev inequality then implies

$$\left(\int \left|\phi u^{\beta/2}\right|^2\right)^{1/\gamma} \leqslant C(1+\beta)^\alpha \int \left(|\nabla\phi|^2+\phi^2\right)u^\beta,$$

where $\gamma = n/(n-2)$ as before. Let $r_1 < r_2 \leqslant r_0$. Choose the cut off function as follows. Let $\phi \in C^1_0(B_{g_\infty}(p,r_2))$ with the property that $\phi \equiv 1$ in $B_{g_\infty}(p,r_1)$ and $|\nabla\phi| \leqslant C/(r_2-r_1)$. Then we obtain

$$\left(\int_{B_g(p,r_1)} u^{\gamma\beta}\right)^{1/\gamma} \leqslant C\frac{(\beta+1)^\alpha}{(r_2-r_1)^2}\int_{B_{g_\infty}(p,r_2)} u^\beta,$$

that is

$$\|u\|_{L^{\gamma\beta}(B_g(p,r_1))} \leqslant \left(C\frac{(\beta+1)^\alpha}{(r_2-r_1)^2}\right)^{1/\gamma}\|u\|_{L^\beta(B_g(p,r_2))}.$$

By Moser iteration technique (exactly as in the proof of Theorem 4.1 in [5]) we get

$$\sup_{B_g(p,1/2)} |\mathrm{Rm}|(g)(x) \leqslant C\biggl(\int_{B_g(p,1)} |\mathrm{Rm}|^{n/2}\biggr)^{2/n},$$

for a uniform constant C. Rescale back to the original metric to get (15). □

4. Topological orbifold structure of a limit

By using a quadratic curvature decay proved in Theorem 9, in this section we will show that we can extract a subsequence of (M_i, g_i) so that it converges to an orbifold in a topological sense. This relies on work by Anderson [1], Bando, Kasue and Nakajima [2] and Tian [10]. Take ϵ_0 to be a small constant from Theorem 9. Define

$$D_i^r = \biggl\{x \in M_i \biggm| \int_{B_{g_i}(x,2r)} |\mathrm{Rm}|^{n/2}\, dV_{g_i} < \epsilon_0\biggr\},$$

and similarly

$$L_i^r = \biggl\{x \in M_i \biggm| \int_{B_{g_i}(x,2r)} |\mathrm{Rm}|^{n/2}\, dV_{g_i} \geqslant \epsilon_0\biggr\}.$$

For each i we can find a maximal $r/2$ separated set, $\{x_k^i\} \in M_i$, so that the geodesic balls $B_{g_i}(x_k^i, r/4)$ are disjoint and $B_{g_i}(x_k^i, r)$ form a cover of M_i. There is a uniform bound on the number of balls m_i^r (centred at x_k^i, with radius r) in L_i^r, independent of i and r, which follows from

$$m_i^r \epsilon_0 \leqslant \sum_{k=1}^{m_i(r)} \int_{B_{g_i}(x_k^i,2r)} |\mathrm{Rm}|^{n/2}\, dV_{g_i} \leqslant m \int_M |\mathrm{Rm}|^{n/2}\, dV_{t_i+t} \leqslant Cm, \tag{20}$$

where m is the maximal number of disjoint balls of radius $r/4$ in M_i contained in a ball of radius $6r$, given by

$$m\kappa(r/4)^n \leqslant \sum_{k=1}^{m} \mathrm{Vol}_{g_i} B_{g_i}(x_k, r/4) \leqslant \mathrm{Vol}_{g_i} B_{g_i}(x, 6r) \leqslant Cr^n.$$

To justify the middle inequality in (20), let K be the number of balls of radius $2r$ intersecting each other. They are contained in a ball of radius $6r$. We have shown that in this ball we can have at most m disjoint balls of radius $r/4$ and therefore $K \leqslant m$.

By Theorem 9 we have that for all $x \in D_i^r$ and $r \leqslant r_0$,

$$\bigl|\mathrm{Rm}(g_i)\bigr|(x) \leqslant \frac{C}{r^2}, \tag{21}$$

for a uniform constant C. This gives the curvatures of g_i being uniformly bounded on D_i^r, which together with volume noncollapsing condition implies a uniform lower bound on injectivity radii. We have seen above there is a uniform upper bound on the number N of points in (M_i, g_i) at which $L^{n/2}$ norm of the curvature concentrates. Assume without loss of generality that $N = 1$. This enables us to assume that $D_i^r = M \backslash B_{g_i}(x_i, 2r)$. Since Kähler Ricci solitons are the solutions of (4) as well, Shi's curvature estimates do apply [8] and therefore by (21),

$$\sup_{M_i \backslash B_{g_i}(x_i, 3r)} \left|D^k \operatorname{Rm}(g_i)\right| \leqslant C(r, k).$$

Denote $G_i^r = M_i \backslash B_{g_i}(x_i, 3r)$. By Cheeger–Gromov convergence theorem, we can extract a subsequence so that (G_i^r, g_i) converges smoothly (uniformly on compact subsets) to a smooth open Kähler manifold G^r with a metric g^r that satisfies a Kähler Ricci soliton equation $g^r - \operatorname{Ric}(g^r) = \partial\bar{\partial} u^r$.

We now choose a sequence $\{r_j\} \to 0$ with $r_{j+1} < r_j/2$ and perform the above construction for every j. If we set $D_i(r_l) = \{x \in M \mid x \in D_i^{r_j}$, for some $j \leqslant l\}$ then we have

$$D_i(r_l) \subset D_i(r_{l+1}) \subset \cdots \subset M_i.$$

For each fixed r_l, by the same arguments as above, each sequence $\{D_i(r_l), g_i\}$ has a smoothly convergent subsequence to a smooth limit $D(r_l)$ with a metric g^{r_l}, satisfying a Kähler Ricci soliton condition. We can now set $D = \bigcup_{l=1}^{\infty} D(r_l)$ with the induced metric g_∞ that coincides with g^{r_l} on $D(r_l)$ and which is smooth on D.

Following Section 5 in [1] we can show there are finitely many points $\{p_i\}$ so that $M_\infty = D \cup \{p_i\}$ is a complete length space with a length function g_∞, which restricts to a Kähler Ricci soliton on D satisfying

$$g_\infty - \operatorname{Ric}(g_\infty) = \partial\bar{\partial} u_\infty,$$

for a Ricci potential u_∞ which is a C^∞ limit of Ricci potentials u_i away from singular points.

To finish the proof of Theorem 2 we still need to show few things:

(a) There is a finite set of points $\{p_1, \ldots, p_N\}$, such that

$$M_\infty = D \cup \{p_i\}$$

is a complete orbifold with isolated singularities $\{p_1, \ldots, p_N\}$.

(b) A limit metric g_∞ on D can be extended to an orbifold metric on M_∞ (denote this extension by g_∞ as well). More precisely, in an orbifold lifting around singular points, in an appropriate gauge, a Kähler Ricci soliton equation of g_∞ can be smoothly extended over the origin in a ball in $\mathbb{C}^{n/2}$.

We will call points $\{p_i\}_{i=1}^N$ *curvature singularities* of M_∞ as in [1]. We want to examine the structure, topological and metric, of M_∞.

The proof that M_∞ has a topological structure of an orbifold is the same as that of [1,2,10] in the case of taking a limit of a sequence of Einstein metrics, so we will just briefly outline the main steps. Without losing generality, assume there is only one singular point, call it p and assume it comes from curvature concentration points $x_i \in M_i$. By Theorem 9 we have that

$$\sup_{M\setminus B_{g_i}(x_i,r)} \bigl|\mathrm{Rm}(g_i)\bigr|(x) \leqslant \frac{C}{r^2}\biggl(\int\limits_{B_{g_i}(x_i,2r)} |\mathrm{Rm}|^{n/2}\,dV_{g_i}\biggr)^{2/n}, \tag{22}$$

which after taking limit on i and using a smooth convergence away from a singular point p yields

$$\sup_{M_\infty\setminus\{p\}} |\mathrm{Rm}|(g_\infty)(x) \leqslant \frac{C}{r(x)^2}\biggl(\int\limits_{B_{g_\infty}(p,2r(x))} |\mathrm{Rm}|^{n/2}\,dV_{g_\infty}\biggr)^{2/n}, \tag{23}$$

where $r(x) = \mathrm{dist}_{g_\infty}(x, p)$. Let $E(r) = \{x \in M_\infty\setminus\{p\} \mid r(x) \leqslant r\}$. Given a sequence $s_i \to 0$, let

$$A(s_i/2, s_i) = \bigl\{x \in M_\infty \mid s_i/2 \leqslant r(x) \leqslant 2s_i\bigr\}.$$

Rescale the metric g_∞ by s_i^{-2}. Then the rescaled Riemannian manifolds $(A(1/2, 1), g_\infty s_i^{-2})$ have sectional curvatures converging to zero by (23). There are uniform bounds on the covariant derivatives $|D^k \mathrm{Rm}|$ of the curvature of metrics $g_\infty s_j^{-2}$ on $A(1/2, 2)$ for the following reasons: An estimate (22) and Shi's curvature estimates give us

$$\sup_{M\setminus B_{g_i}(x_i,2r)} \bigl|D^k \mathrm{Rm}(g_i)\bigr|(x) \leqslant \frac{C}{r^{k+2}}.$$

Letting $i \to \infty$ we get

$$\bigl|D^k \mathrm{Rm}\bigr|(g_\infty)(x) \leqslant \frac{C_1}{r(x)^{k+2}}, \tag{24}$$

for all $x \in M_\infty\setminus\{p\}$. This tells us there are uniform bounds on the covariant derivatives $|D^k \mathrm{Rm}|$ of the curvature of the metric $g_\infty s_i^{-2}$ on $A(1/2, 2)$.

As in [1] and [10] we can get a uniform bound, independent of r, on a number of connected components in $E(r)$. It follows now that a subsequence $(A(1/2, 2), g_\infty s_i^{-2})$ converges smoothly to a flat Kähler manifold $A_\infty(1/2, 2)$ with a finite number of components. If we repeat this process for $A(s_i/k, ks_i)$, for any given k, passing to a diagonal subsequence, it gives rise to a flat Kähler manifold A_∞. As in [1,2,10] we can show that each component of A_∞ is a cone on a spherical space form $S^{n-1}(1)/\Gamma$ and that every component is diffeomorphic to $(0, r) \times S^{n-1}/\Gamma$. We will call (M_∞, g_∞) a *generalized orbifold*.

So far we have proved the following proposition.

Proposition 11. *Let (M_i, g_i) be a sequence of compact Kähler Ricci solitons, with $c_1(M_i) > 0$, such that $g_i \in \mathcal{S}_n(C_2, A, V)$ and such that there is a uniform constant C, so that*

$$\int\limits_{M_i} \bigl|\mathrm{Rm}(g_i)\bigr|^{n/2}\,dV_{g_i} \leqslant C.$$

Then there is a subsequence so that (M_i, g_i) converges in the sense of part (a) *of Definition* 4 *to a compact generalized orbifold (M_∞, g_∞) with finitely many singularities. Convergence is smooth outside those singular points and g_∞ can be extended to a C^0 metric in an orbifold sense* (*in the corresponding liftings around singular points*).

5. A smooth metric structure of a limit orbifold M_∞ for $n \geqslant 6$

In this section we will always assume $n \geqslant 6$. We will show that a limit metric g_∞ can be extended to an orbifold metric in C^∞ sense. More precisely, let p be a singular point with a neighbourhood $U \subset M_\infty$. Let U_β be a component of $U\backslash\{\text{singular points}\}$. Recall that each U_β is covered by $\Delta_r^* = \Delta_r\backslash\{0\}$. We will show that in an appropriate gauge, the lifting of g_∞ (around singular points) can be smoothly extended to a smooth metric in a ball Δ_r in $\mathbb{C}^{n/2}$. Metric g_∞ comes in as a limit of Kähler Ricci solitons $g_i \in \mathcal{S}_n(C_2, A, V)$. A Sobolev inequality with a uniform Sobolev constant S holds for all g_i. We will show that a Sobolev inequality with the same Sobolev constant S holds for g_∞ as well.

Lemma 12. *There is r_0 so that for every $r \leqslant r_0$,*

$$\left(\int_B v^{\frac{2n}{n-2}}\,dV_{g_\infty}\right)^{\frac{n-2}{n}} \leqslant S\int_B |\nabla v|^2\,dV_{g_\infty}, \tag{25}$$

for every $v \in C_0^1(B\backslash\{p\})$, where $B = B_{g_\infty}(p, r)$.

Proof. Take r_0 such that $\mathrm{Vol}_{g_\infty} B_{g_\infty}(p, r_0) \leqslant V r_0^n < \epsilon_0$, where ϵ_0 is a small constant from Theorem 9. Let $v \in C_0^1(B\backslash\{p\})$ and let $\mathrm{supp}(v) = K \subset B\backslash\{p\}$. By the definition of convergence, there exist diffeomorphisms ϕ_i from the open subsets of $M\backslash\{x_i\}$ to the open subsets of $M\backslash\{p\}$ that contain K, such that every diffeomorphism ϕ_i maps some compact subset K_i onto K, where K_i is contained in $B_i = B_{g_i}(x_i, r)$, for some sufficiently large i (because of the uniform convergence of metrics on compact subsets). We have that $\tilde{g_i} = (\phi_i^{-1})^* g_i$ converge uniformly and smoothly on K to g_∞.

Let $F_i = \phi_i^*(v)$. Then, $\mathrm{supp}\, F_i \subset K_i \subset B(x_i, r)\backslash\{x_i\}$. Let $\{\eta_i^k\}$ be a sequence of cut-off functions, such that $\eta_i^k \in C_0^1(B_i\backslash\{x_i\})$ and $\eta_i^k \to 1$ $(k \to \infty)$ $\forall i$, and:

$$\int_{B_i} \left|D\eta_i^k\right|^2 \to 0 \quad (k \to \infty).$$

$\eta_i^k F_i$ is a function of compact support in B_i. Then by Sobolev inequality:

$$\left(\int_{B_i} \left|\eta_i^k F_i\right|^{\frac{2n}{n-2}}\,dV_{g_i}\right)^{\frac{n-2}{n}} \leqslant S\int_{B_i} \left|D\left(\eta_i^k F_i\right)\right|^2\,dV_{g_i}.$$

We can bound F_i with some constant C_i (as a continuous function on a compact set), and therefore:

$$\int_{B_i} \left|D\left(\eta_i^k F_i\right)\right|^2 \leqslant S\left(\int_{B_i} \left|D\eta_i^k\right|^2 C_i + \int_{B_i} |DF_i|^2\left(\eta_i^k\right)^2\right).$$

Let k tend to ∞. Then we get:

$$\left(\int_{B_i\backslash\{x_i\}} |F_i|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leqslant C \int_{B_i\backslash\{x_i\}} |DF_i|^2.$$

Since $\operatorname{supp} F_i \subset K_i$, after changing the coordinates via map ϕ_i we get:

$$\left(\int_K |v|^{\frac{2n}{n-2}}\, dV_{\tilde{g}_i}\right)^{\frac{n-2}{n}} \leqslant S\int_K |Dv|^2\, dV_{\tilde{g}_i}.$$

Metrics $\{\tilde{g}_i\}$ converge uniformly on K to g_∞, so if we let i tend to ∞ in the above inequality, keeping in mind that $\operatorname{supp} v = K \subset B\backslash\{p\}$, we get that

$$\left(\int_B |v|^{\frac{2n}{n-2}}\, dV_{g_\infty}\right)^{\frac{n-2}{n}} \leqslant S\int_B |Dv|^2\, dV_{g_\infty}. \qquad \square$$

Remark 13. Observe that (25) also holds for $v \in W^{1,2}(B)$ (by similar arguments as in [2]). Namely, let $L_k(t)$ be

$$L_k(t) = \begin{cases} k, & \text{for } t \geqslant k,\\ t, & \text{for } |t| < k,\\ -k, & \text{for } t \leqslant -k.\end{cases}$$

Then,

$$\left\{\int_B \left|L_k(v)\right|^{2\gamma}\right\}^{1/\gamma} \leqslant S\int_B \left|\nabla L_k(v)\right|^2 = S\int_{|v|<k} |\nabla v|^2.$$

Letting $k \to \infty$, by Fatou's lemma we get (25) for $v \in W^{1,2}(B)$.

5.1. Curvature bounds in punctured neighbourhoods of singular points

Choose $r_0' = 2r_0$ as in Lemma 12. Decrease r_0' if necessary so that

$$\int_{B_{g_\infty}(p,2r_0)} |\mathrm{Rm}|^{n/2}\, dV_{g_\infty} < \epsilon,$$

where ϵ is chosen to be small. Since g_∞ is a limit metric of a sequence of Kähler Ricci solitons whose curvatures satisfy (12), we get that $\mathrm{Rm} = \mathrm{Rm}(g_\infty)$ also satisfies

$$\Delta|\mathrm{Rm}||\mathrm{Rm}| \geqslant -\theta/2\big|\nabla|\mathrm{Rm}|\big|^2 - C(\theta)|\mathrm{Rm}|^2 - C|\mathrm{Rm}|^3, \tag{26}$$

for small $\theta \in (0,1)$. Our goal is to show that the curvature of g_∞ is uniformly bounded on $B_{g_\infty}(p, 2r_0)\backslash\{p\}$. We will use this curvature bound to show a smooth extension of a lifting of an orbifold metric over the origin in C^n. Denote by $u = |\mathrm{Rm}|$. Function u is then a nonnegative solution satisfying

$$u\Delta u \geqslant -\theta/2|\nabla u|^2 - Cu^2 - Cu^3. \tag{27}$$

This is a special case of more general inequality

$$u\Delta u \geqslant -\theta/2|\nabla u|^2 - Cfu^2, \tag{28}$$

where $f \in L^{n/2}$. By Fatou's lemma we also have that

$$\int\limits_{M_\infty} |\mathrm{Rm}|^{n/2}\, dV_{g_\infty} \leqslant \liminf_{i\to\infty} \int\limits_{M} |\mathrm{Rm}|^{n/2}\, dV_{g_i} \leqslant C.$$

Remember that we are treating the case $n \geqslant 6$ which allows us to adopt the approach of Sibner in [9]. Our goal is to show that $u \in L^p(B)$, for some $p > n/2$, because it will give us a uniform bound on the curvature of g_∞ away from a singular point p. The proof of the following lemma is similar to the proof of Lemma 2.1 in [9].

Lemma 14. *Let $u \geqslant 0$ be C^∞ in $M_\infty\backslash\{p\}$ and satisfy there* (27)*, with $u \in L^{n/2}$. If $u \in L^{2nq_0/(n-2)} \cap L^{2q}$, with $q_0 > 0$ fixed, then $\nabla u^q \in L^2$ and in a sufficiently small ball B, for all $\eta \in C_0^\infty(B)$,*

$$\int\limits_B \eta^2\big|\nabla u^q\big|^2 \leqslant C\int\limits_B |\nabla\eta|^2 u^{2q}.$$

Proof. As in [9] we will choose a particularly useful test function. Let

$$F(u) = \begin{cases} u^q, & \text{for } 0 \leqslant u \leqslant l, \\ \frac{1}{q_0}(ql^{q-q_0}u^{q_0} + (q_0 - q)l^q), & \text{for } l \leqslant u, \end{cases}$$

and

$$F_1(u) = \begin{cases} u^{q-1}, & \text{for } 0 \leqslant u \leqslant l, \\ \frac{1}{q_0}(ql^{q-q_0}u^{q_0-1} + \frac{(q_0-q)l^q}{u}), & \text{for } l \leqslant u. \end{cases}$$

Then (see [9]) the following inequalities are satisfied:

$$F \leqslant \frac{q}{q_0} l^{q-q_0} u^{q_0}, \tag{29}$$

$$u F F' \leqslant q F^2, \tag{30}$$

$$(F F')' \geqslant C' F'^2, \quad C' > 0, \tag{31}$$

where the last inequality fails if $q_0 \leqslant 1/2$ (that is the reason we have assumed $n > 4$ at the moment). Let $\eta \in C_0^\infty(B)$, for a sufficiently small ball B and $\bar{\eta} = 0$ in a neighbourhood of p. If ξ is a test function, from (27) we have

$$\int \nabla u \nabla (u\xi) \leqslant \theta/2 \int |\nabla u|^2 \xi + C \int \left(u^2 + u^3\right)\xi.$$

In particular, choose ξ to be $(\eta\bar{\eta})^2 F_1(u) F'(u)$. Integrating by parts, using (31), we get

$$\begin{aligned}
&\int \left|\nabla F(u)\right|^2 (\eta\bar{\eta})^2 - \frac{\theta}{C} \int |\nabla u|^2 (\eta\bar{\eta})^2 F_1 F' \\
&\quad \leqslant C_1 \int \nabla u F F' (\eta\bar{\eta}) \nabla(\eta\bar{\eta}) + C_1 \int (u+1) u F F' (\eta\bar{\eta})^2,
\end{aligned}$$

$$\begin{aligned}
C_1 \int \nabla u F F' (\eta\bar{\eta}) \nabla(\eta\bar{\eta}) &= C_1 \int \nabla F(u) F (\eta\bar{\eta}) \nabla(\eta\bar{\eta}) \\
&\leqslant 1/4 \int \left|\nabla F(u)\right|^2 (\eta\bar{\eta})^2 + C_2 \int F^2 \left|\nabla(\eta\bar{\eta})\right|^2.
\end{aligned}$$

Using Lemma 12 and (30) we get

$$\begin{aligned}
C_1 \int (u+1) u F F' (\eta\bar{\eta})^2 &\leqslant C_1 q \int (u+1) F^2 (\eta\bar{\eta})^2 \\
&\leqslant C_1 q \left\{ \int (u+1)^{n/2} \right\}^{2/n} \left\{ \int (F\eta\bar{\eta})^{\frac{2n}{n-2}} \right\}^{\frac{n}{n-2}} \\
&\leqslant C_1 q S \|u+1\|_{L^{n/2}(B)} \left\|\nabla(\eta\bar{\eta})\right\|_2^2,
\end{aligned}$$

where S is a Sobolev constant. Since $u + 1 \in L^{n/2}$, we can choose B small so that

$$C_1 q S \|u+1\|_{L^{n/2}(B)} < 1/4.$$

Then,

$$\int \left|\nabla F(u)\right|^2 (\eta\bar{\eta})^2 - \frac{\theta}{C_3} \int |\nabla u|^2 (\eta\bar{\eta})^2 F_1 F' \leqslant C_4 \int F^2 \left|\nabla(\eta\bar{\eta})\right|^2. \tag{32}$$

Choose a sequence $\eta_k \to 1$ on B with $\int |\nabla \eta_k|^n \to 0$ as $k \to \infty$. The term we have to estimate is

$$\int \eta^2 |\nabla \eta_k|^2 F^2 \leqslant C(l) \int |\nabla \eta_k|^2 u^{2q_0} \leqslant C(l) \left\{ \int |\nabla \eta_k|^n \right\}^{2/n} \left\{ \int u^{\frac{2nq_0}{n-2}} \right\}^{(n-2)/n},$$

which tends to zero as $k \to \infty$, since the last factor on the right is bounded. If we let $k \to \infty$ in (32), we get

$$\int \left|\nabla F(u)\right|^2 \eta^2 - \frac{\theta}{C_3} \int |\nabla u|^2 \eta^2 F_1 F' \leqslant C_4 \int |\nabla \eta|^2 F^2. \tag{33}$$

We will see later we may assume $q_0 = 1$ and $q_0 \leqslant q$. Choose small θ so that:

- for $u \leqslant l$,

$$\left|\nabla F(u)\right|^2 - \frac{\theta}{C_3} F_1 F' |\nabla u|^2 = F' |\nabla u|^2 \left(F' - \frac{\theta}{C_3} F_1\right) = u^{q-1} F' |\nabla u|^2 \left(q - \frac{\theta}{C_3}\right) \geqslant 0,$$

- and for $u \geqslant l$,

$$\left|\nabla F(u)\right|^2 - \frac{\theta}{C_3} F_1 F' |\nabla u|^2 = F' |\nabla u|^2 \left\{ q u^{q_0 - 1} l^{q - q_0} \left(1 - \frac{\theta}{C_3 q_0}\right) - \frac{(q_0 - q) l^q}{u} \right\} \geqslant 0.$$

This implies

$$\int_{u \leqslant l} \left|\nabla F(u)\right|^2 \eta^2 \left(1 - \frac{\theta}{C_3 q}\right) \leqslant C_4 \int |\nabla \eta|^2 F^2. \tag{34}$$

For every l we define F. Since for $u \geqslant l$, we have that $\frac{1}{q_0}(q l^{q-q_0} u^{q_0} + (q_0 - q) l^q) \leqslant \frac{q}{q_0} u^q$ and since $u \in L^{2q}$, for every $\epsilon > 0$ there is δ so that whenever $\mathrm{Vol}(E) < \delta$, for every l we have $\int_E |\nabla \eta|^2 F^2 < \epsilon$. Moreover, there is l_0 so that for all $l \geqslant l_0$, we have $\mathrm{Vol}(\{u \geqslant l\}) \leqslant \frac{\int_B u^{2q}}{l^q} < \delta$, which implies

$$\int |\nabla \eta|^2 F^2 = \int_{u \leqslant l} |\nabla \eta|^2 F^2 + \int_{u \geqslant l} |\nabla \eta|^2 F^2 < \int_{u \leqslant l} |\nabla \eta|^2 u^{2q} + \epsilon.$$

Since $F(u) \to u^q$ as $l \to \infty$, letting $l \to \infty$ in (34) we get

$$\int \eta^2 \left|\nabla u^q\right|^2 \leqslant C_5 \int |\nabla \eta|^2 u^{2q} + \epsilon.$$

Since $\epsilon > 0$ can be arbitrarily small, we get

$$\int \eta^2 \left|\nabla u^q\right|^2 \leqslant C_5 \int |\nabla \eta|^2 u^{2q}. \qquad \square$$

Lemma 15. *Let u be a nonnegative function as above. Then $u \in L^p$, for some $p > n/2$.*

Proof. Since $u = |\mathrm{Rm}(g_\infty)| \in L^{n/2}$ and $n \geqslant 6$ (we have assumed the real dimension $n > 4$), we can choose $q_0 = 1$ and $q = n/4$. Since u is a nonnegative solution of (27), applying Lemma 14 to u, we find that $\nabla u^{n/4} \in L^2(B)$. By Remark 13, we can apply Sobolev inequality to $u^{n/4}$ to conclude that $u \in L^p$ with $p = \frac{n}{2}(\frac{n}{n-2}) > \frac{n}{2}$. $\square$

Since $\mathrm{Vol}_{g_\infty}(M_\infty) < \infty$, by the previous lemma, $u \in L^p(B)$, for $p \in (0, \frac{n^2}{2(n-2)}]$. Take $q_0 = 1$, $q \in [n/2, \frac{n}{4}\frac{2n}{n-2}]$ and repeat the proof of Lemma 14 to get $\nabla u^q \in L^2(B)$ for all such q. By Remark 13 we have $u \in L^s(B)$, for $s \in [\frac{n}{4}\frac{2n}{n-2}, \frac{n}{4}(\frac{2n}{n-2})^2]$. If we keep on repeating this, at the kth step we get $\nabla u^q \in L^2$ for $q \in (0, \frac{n}{4}(\frac{2n}{n-2})^k]$ and $u \in L^q(B)$ for $q \in (0, \frac{n}{4}(\frac{2n}{n-2})^{k+1}]$. Since $(\frac{2n}{n-2})^k \to \infty$ as $k \to \infty$, we can draw the following conclusion.

Lemma 16. *If we adopt the notation from above, we have $u \in L^q(B)$ and $\nabla u^q \in L^2(B)$ for all q.*

Remark 17. We could get the same conclusion for nonnegative functions u satisfying (28) with $f \in L^{n/2}$ (in the case of $u = |\mathrm{Rm}(g_\infty)|$, $f = u + 1$).

The previous lemma helps us get the uniform bound on the curvature of g_∞ in a punctured neighbourhood of a singular point p, that is, we have the following proposition in the case $n > 4$.

Proposition 18. *There is a uniform constant C and $r_0 > 0$ so that*

$$\sup_{B\backslash\{p\}} \left|\mathrm{Rm}(g_\infty)\right|(x) \leqslant \frac{C}{r_0^2},$$

where $B = B_{g_\infty}(p, r_0)$.

Proof. Combining Lemma 16 and Remark 13, for any cut off function η with compact support in B and any q we have a Sobolev inequality

$$\left(\int \left|\eta u^{q/2}\right|^{2\gamma}\right)^{1/\gamma} \leqslant C \int \left|\nabla\left(\eta u^{q/2}\right)\right|^2,$$

with a uniform constant C. The rest of the proof is the same as the proof of Theorem 9 in the case of a smooth shrinking Kähler Ricci soliton. We get

$$\sup_{B_{g_\infty}(p,r/2)\backslash\{p\}} \left|\mathrm{Rm}(g_\infty)\right|(x) \leqslant \frac{C}{r_0^2}. \qquad \square$$

5.2. Smoothing property of Kähler Ricci solitons

In Section 4 we have showed that g_∞ extends to a C^0-metric on M_∞ in the sense that each singular point $p_i \in M_\infty$ has a neighbourhood that is covered by a smooth manifold, diffeomorphic to a punctured ball $\Delta_r^* \subset \mathbb{C}^{n/2}$. If we denote by ϕ_i those diffeomorphisms and by π_i the covering maps, then the pull-back metric $\phi_i^* \circ \pi_i^*(g_\infty)$ extends to a C^0-metric on the ball Δ_r. We know that g_∞ satisfies a soliton equation away from orbifold points. Note that the metric $\phi_i^*\pi^* g_\infty$ is a Kähler Ricci soliton in Δ_r, outside the origin. Our goal is to show that g_∞ extends to a C^∞-metric on Δ_r. That implies $\phi_i^* \circ \pi_i^*(g_\infty)$ satisfies the soliton equation on Δ_r, that is g_∞ is a soliton metric in an *orbifold sense* (see Definition 4).

Using Proposition 18 in the case $n \geqslant 6$ and Proposition 22 in the case $n = 4$, and harmonic coordinates constructed in [4], in the same way as in Lemma 4.4 in [10] and in the proof of

Theorem 5.1 in [2] we can show that if r is sufficiently small, there is a diffeomorphism ψ of Δ_r^* such that ψ extends to a homeomorphism of Δ_r and

$$\psi^*(g_\infty)_{i\bar{j}}(x) - \delta_{i\bar{j}} = O\big(|x|^2\big),$$
$$\partial_k \psi^*(g_\infty)_{ij}(x) = O\big(|x|\big).$$

This means $\psi^* g_\infty$ is of class $C^{1,1}$ on Δ_r, that is, there are some coordinates in a covering of a singular point of M_∞ in which g_∞ extends to a $C^{1,1}$-metric (we may assume g_∞ is $C^{1,1}$ for further consideration).

Lemma 19. *Metric g_∞ is actually C^∞ on Δ_r.*

Proof. In Section 3 we have showed that Ricci potentials u_i satisfy the following equation:

$$2\Delta u_i - |\nabla u_i|^2 + R(g_i) + u_i - n = \mu(g_i, 1/2),$$

with $\Delta u_i = n/2 - R(g_i)$ and therefore,

$$\Delta u_i = |\nabla u_i|^2 - u_i + n/2 + \mu(g_i, 1/2). \tag{35}$$

We have showed that metrics $\{g_i\}$ uniformly and smoothly converge to a metric g_∞ on compact subsets of $M_\infty \backslash \{p\}$ (we are still assuming there is only one singular point, a general case is treated in the same way). The uniform C^1 bounds on u_i (see Section 3) and a uniform bound on $R(g_i)$, together with condition $\mu(g_i, 1/2) \geqslant A$ give

$$A \leqslant \mu(g_i, 1/2) \leqslant \tilde{C},$$

for some uniform constant $\tilde{C}$. We can extract a subsequence of a sequence of converging metrics g_i so that $\lim_{i\to\infty} \mu(g_i, 1/2) = \mu_\infty$. If we let $i \to \infty$ in (35) we get

$$\Delta u_\infty = |\nabla u_\infty|^2 - u_\infty - \mu_\infty - n/2, \tag{36}$$

with $(u_\infty)_{ij} = 0$ away from a singular point p. Proposition 18 gives uniform bounds on $|\mathrm{Rm}(g_\infty)|$ on $M_\infty \backslash \{p\}$ and therefore,

$$\sup_{M_\infty \backslash \{p\}} |\nabla \bar{\nabla} u_\infty|_{g_\infty} \leqslant C,$$

for a uniform constant C. Since we also have that $(u_\infty)_{ij} = (u_\infty)_{\bar{i}\bar{j}} = 0$, this together with $\sup_{M_\infty \backslash \{p\}} |u_\infty|_{C^1(M_\infty \backslash \{p\})} \leqslant C$ (which comes from $|u_i|_{C^1} \leqslant C$) yields

$$\sup_{M_\infty \backslash \{p\}} |u_\infty|_{C^2} \leqslant C. \tag{37}$$

Since g_∞ is $C^{1,1}$ in Δ_r, and since for any two points $x, y \in \Delta_r^*$ such that a line $\overline{xy}$ does not contain the origin, due to (37), we have $|\nabla u_\infty(x) - \nabla u_\infty(y)| \leqslant C\, \mathrm{dist}_{g_\infty}(x, y)$ (the set of such x and y is dense in M_∞). Therefore, ∇u_∞ extends to the origin in Δ_r. Moreover, $u_\infty \in C^{1,1}(\Delta_r)$.

Take the harmonic coordinates Φ for g_∞ (see [3]) in Δ_r. Outside the origin, Φ is smooth and $h = \Phi^* g_\infty$ satisfies

$$\Delta h = 2h - \partial\bar{\partial} u_\infty. \tag{38}$$

Since $g_\infty \in C^1(\Delta_r)$ and $u_\infty \in C^{1,1}(\Delta_r)$, the right-hand side of (36) is in $C^{0,1}(\Delta_r)$. By elliptic regularity this implies $u_\infty \in C^{2,\alpha}(\Delta_r)$, for some $\alpha \in (0,1)$. We have that $u_\infty \in C^{2,\alpha}(\Delta_r)$ and g_∞ is of class $C^{1,1}(\Delta_r)$ and therefore by results of De Turck and Kazdan in [3], u_∞ is at least $C^{1,1}$ and g_∞ is of class $C^{1,1}$ in harmonic coordinates in Δ_r. We will write g_∞ for $\Phi^* g_\infty$ and from now on when we mention regularity, or being of class $C^{k,\alpha}$, we will assume harmonic coordinates. The right-hand side of (36) is of class $C^{0,1}(\Delta_r)$, so by elliptic regularity, u_∞ is of class $C^{2,\alpha}(\Delta_r)$. By elliptic regularity applied to (38), we get g_∞ is $C^{2,\alpha}(\Delta_r)$. From (36) we get u_∞ being of class $C^{3,\alpha}(\Delta_r)$, since the right-hand side of (36) is in $C^{1,\alpha}(\Delta_r)$. Now again by (38), g_∞ is of class $C^{3,\alpha}$ in Δ_r.

If we keep repeating the argument from above, we will obtain that g_∞ is of class C^k in Δ_r, for any k, that is, there are coordinates ψ in Δ_r (disc Δ_r covers a neighbourhood of an orbifold point in M_∞), such that $\psi^*\pi^* g_\infty$ in those coordinates extends to a C^∞-metric on Δ_r, where π is just a covering map. In particular, the Kähler Ricci soliton equation of $\psi^*\pi^* g_\infty$ holds in all Δ_r. $\square$

6. A smooth orbifold singularity in the case $n = 4$

In this section we deal with the case $n = 4$, that is, we want to prove Theorem 3. The first four sections and the subsection apply to a four-dimensional case, but a different approach has to be taken when one wants to prove the curvature of a limit metric g_∞ is uniformly bounded away from isolated singularities. We will use the same notation from the previous sections. To prove that g_∞ extends smoothly to a smooth orbifold metric, satisfying the Kähler Ricci soliton equation in a lifting around each singular point we will use Uhlenbeck's theory of removing singularities of Yang–Mills connections in a similar way Tian used it in [10].

Assume M_∞ has only one singular point p. Let U be a small neighbourhood of p and let U_β be a component of $U\backslash\{p\}$. Recall that U_β is covered by $\Delta_r^* \subset C^2$ with a covering group Γ_β isomorphic to a finite group in $U(2)$ and $\pi_\beta^* g_\infty$ extends to a C^0 metric on the ball Δ_r, where π_β is a covering map. In order to prove g_∞ extends to a smooth metric in a covering, we first want to prove the boundness of a curvature tensor $\mathrm{Rm}(g_\infty)$. The proof is similar to that for Yang–Mills connections in [12] with some modifications. Considerations based on similar analysis can be found in [10] and [11]. We will just consider Δ_r^* as a real 4-dimensional manifold.

In Section 4 we saw there is a gauge ϕ so that by estimates (23) and (24),

$$\|dg_{ij}\|_{g_F}(x) \leqslant \frac{\epsilon(r(x))}{r(x)},$$

$$\left\| d\left(\frac{\partial g_{ij}}{\partial x_k}\right)\right\|_{g_F} \leqslant \frac{\epsilon(r(x))}{r(x)^2},$$

$$\left\| d\left(\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}\right)\right\|_{g_F} \leqslant \frac{\epsilon(r(x))}{r(x)^3},$$

in $B_{g_\infty}(p,r)$, where d is the exterior differential on $\mathrm{C}^2 = \mathrm{R}^4$ and $\|\cdot\|_{g_F}$ is the norm on $T^1\mathrm{R}^4$ with respect to the euclidean metric g_F, and g stands for $\phi^* g_\infty$.

Let $\tilde{A}$ be a connection form uniquely associated to a metric g_∞ on Δ_r^*, that is $\tilde{D} = d + \tilde{A}$ is the covariant derivative with respect to g_∞. We can view $\tilde{A}$ as a function in $C^{1,\alpha}(B_0(p,r), so(4) \times \mathrm{R}^4)$ for $\alpha \in (0,1)$. The following lemma is essentially due to Uhlenbeck [12], but the form in which we will state it below can be found in [10]. It applies to our case as well, since we also have an ϵ-regularity theorem as in [10].

Lemma 20. *Let r be sufficiently small. There is a gauge transformation u in $C^\infty(B(p,2r), so(4))$ such that if $D = e^{-u}\tilde{D}e^u = d + A$, then $d^*A = 0$ on $B(p,2r)$, $d_\psi^* A_\psi = 0$ on $\partial B(p,2r)$, where d^*, d_ψ^* are the adjoint operators of the exterior differentials on $B(p,2r)$ or $\partial B(p,2r)$ with respect to g, respectively. We also have that*

$$\sup_{\Delta(r,2r)} \big(\|A\|_g(x)\big) \leqslant \frac{\epsilon(r)}{r},$$

where $\epsilon(r) \to 0$ as $r \to 0$.

We have also the following estimates due to Tian (Lemma 4.2 in [10]).

Lemma 21. *Let A be the connection given in Lemma 20. For small r we have*

$$\sup_{\Delta(r,2r)} \|A\|_g(x) \leqslant Cr \sup_{\Delta(r,2r)} \|R_A\|_g(x),$$

$$\int_{\Delta(r,2r)} \|A\|_g^2(x)\,dV_g \leqslant Cr^2 \int_{\Delta(r,2r)} \|R_A\|_g^2\,dV_g.$$

Proposition 22. *There exist $0 < \delta < 1$ and $r > 0$ such that*

$$\big|\mathrm{Rm}(g_\infty)\big|(x) \leqslant \frac{C}{r(x)^\delta},$$

where $r(x) = \mathrm{dist}_{g_\infty}(x,p)$ for $x \in \Delta_r^*$.

Proof. The proof is the same as that of Proposition 4.7 in [12]. Let $r_1 = r/2, \ldots, r_i = (r_{i-1})/2, \ldots$. Let A_i be a connection given by Lemma 20. As in Lemma 4.3, in [10], if we put $\Omega_i = \Delta(r_i, r_{i-1})$, we have

$$\begin{aligned}\int_{\Omega_i} \|R_{A_i}\|_{g_\infty}^2\,dV_{g_\infty} = &-\int_{\Omega_i} \big\langle [A_i, A_i], R_{A_i}\big\rangle_{g_\infty} dV_{g_\infty} - \int_{\Omega_i} \big\langle A_i, D_i^* R_{A_i}\big\rangle_{g_\infty} dV_{g_\infty} \\ &- \int_{S_i} \big\langle A_{i\psi}, (R_{A_i})_{r\psi}\big\rangle_{g_\infty} + \int_{S_{i-1}} \big\langle A_{i\psi}, (R_{A_i})_{r\psi}\big\rangle_{g_\infty},\end{aligned}$$

where $S_i = \partial\Delta_{r_i}$, $D_i = d + A_i$. If we sum those identities over i, we get

$$\int_{\Delta(r,2r)} \|\mathrm{Rm}(g_\infty)\|^2 \, dV_{g_\infty} = -\sum_i \int_{\Omega_i} \langle R_{A_i}, [A_i, A_i] \rangle \, dV_{g_\infty} - \sum_i \int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle \, dV_{g_\infty} + \int_{\partial \Delta_r} \langle A_{1\psi}, (R_{A_1})_{r\psi} \rangle.$$

To get the conclusion of Proposition 22 we proceed in exactly the same way as in [10]. We do not have a Yang–Mills or an Einstein condition, so we have to use the Ricci soliton equation to estimate a term $\int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle \, dV_{g_\infty}$ that appears below. We know g_∞ satisfies

$$(g_\infty)_{i\bar{j}} - R_{i\bar{j}} = u_{i\bar{j}},$$

with $u_{ij} = u_{\bar{i}\bar{j}} = 0$. By Bianchi identity we have $D^* \mathrm{Rm} = d^\nabla \mathrm{Ric}$. Since $\nabla_k R_{i\bar{j}} = -u_{i\bar{j},k} = -R_{i\bar{j}k\bar{l}}$, and $|\nabla u| \leqslant C$ on Δ_r^*, by Lemma 21 we have

$$\begin{aligned}\int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle \, dV_{g_\infty} &\leqslant \Big(\int_{\Omega_i} \|A_i\|_{g_\infty}^2 \Big)^{1/2} \Big(\int_{\Omega_i} |d^\nabla \mathrm{Ric}|^2 \Big)^{1/2} \\ &\leqslant C r_i \Big(\int_{\Omega_i} |\mathrm{Rm}|^2 \, dV_{g_\infty} \Big)^{1/2} \Big(\int_{\Omega_i} |\mathrm{Rm}|^2 \, dV_{g_\infty} \Big)^{1/2} = C r_i \int_{\Omega_i} |\mathrm{Rm}|^2 \, dV_{g_\infty}.\end{aligned}$$

This yields

$$\sum_i \int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle \, dV_{g_\infty} \leqslant C r \int_{\Delta(r,2r)} |\mathrm{Rm}|^2 \, dV_{g_\infty}.$$

Similarly as in [10] we get

$$|\mathrm{Rm}|_{g_\infty}(x) \leqslant \frac{C}{r(x)^\delta},$$

for $x \in \Delta_r^*$; for sufficiently small r and some $\delta \in (0,1)$. □

Section 6 applies to the case $n = 4$ as well and that concludes the proof of Theorem 3.

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